数理解析研究所講究録
第 1911 巻 2014 年 18-33

Hasse-Weil zeta functions of SL_2-character varieties of 3-manifolds

Shinya Harada
Graduate School of Information Science and Engineering, Tokyo Institute of Technology

1 Introduction

The $SL_2(\mathbb{C})$-character varieties of a 3-manifold is known to be a powerful tool for the study of 3-manifolds. It is showed by the work of Culler-Shalen in [4] that they can be used to construct essential surfaces in the 3-manifolds. $SL_2(\mathbb{C})$-character varieties are algebraic varieties defined by finite number of polynomials with rational coefficients (precisely they are affine algebraic sets). Later another invariant, A-polynomials are defined in [3] for 3-manifolds with one cusp, which enabled us to study the essential surfaces coming from the $SL_2(\mathbb{C})$-character varieties more directly.

Hasse-Weil zeta functions in the title are natural generalization of the Dedekind zeta functions for algebraic varieties over number fields. As we could see in the class number formula for the Dedekind zeta functions and the Birth and Swinnerton-Dyer conjecture for elliptic curves, we expect that they inherit geometric and number theoretic properties of the varieties.

In this note we consider Hasse-Weil zeta functions of the $SL_2(\mathbb{C})$-character varieties of certain hyperbolic 3-manifolds and the A-polynomials of torus knots, and study what kind of topological information appears in the description of the zeta functions.

2 Hasse-Weil zeta functions of polynomials

2.1 Local zeta function

Let p be a prime number and $\mathbb{F}_{p^n}$ the finite field with p^n elements. For given finite number of polynomials $f_1, \cdots, f_r \in \mathbb{Z}[X_1, \cdots, X_m]$ we denote by $V(f_1, \cdots, f_r; \mathbb{F}_{p^n})$ the set of $\mathbb{F}_{p^n}$-rational points of $f_1, \cdots, f_r$:

$$V(\mathbb{F}_{p^n}) := V(f_1, \cdots, f_r; \mathbb{F}_{p^n}) = \left\{(a_1, \cdots, a_m) \in (\mathbb{F}_{p^n})^m \middle| f_1(a_1, \cdots, a_m) = \cdots = f_r(a_1, \cdots, a_m) = 0\right\}.$$

Then the local (congruence) zeta function of V at p is defined by

$$Z(V, p, T) := \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{p^n})}{n} T^n\right) \in \mathbb{Q}[[T]].$$

Here

$$\exp(T) := \sum_{n=0}^{\infty} \frac{1}{n!} T^n, \quad \log\left(\frac{1}{1-T}\right) = \sum_{n=1}^{\infty} \frac{1}{n} T^n.$$

Example 2.1. Consider the case $f = X \in \mathbb{Z}[X]$ and $V = V(f)$. Then it is clear that $V(\mathbb{F}_{p^n}) = \{0\}$. Therefore $\#V(\mathbb{F}_{p^n}) = 1$ and we have

$$Z(V, p, T) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} T^n\right) = \frac{1}{1-T}.$$

Example 2.2. When $f = X^2 + 1 \in \mathbb{Z}[X]$. In this case

$$\#V(f)(\mathbb{F}_{p^n}) = \begin{cases} 1, & p = 2, \\ 2, & p \neq 2, \left(\frac{-1}{p}\right) = 1 \text{ or } \left(\frac{-1}{p}\right) = -1, \quad n \equiv 0 \pmod 2, \\ 0, & p \neq 2, \left(\frac{-1}{p}\right) = -1, \quad n \equiv 1 \pmod 2. \end{cases}$$

Here $\left(\frac{-1}{p}\right)$ is the Legendre symbol. Thus we have

$$Z(V, p, T) = \begin{cases} 1/(1-T), & p = 2, \\ 1/(1-T)^2, & p \neq 2, \left(\frac{-1}{p}\right) = 1, \\ 1/(1-T^2), & p \neq 2, \left(\frac{-1}{p}\right) = -1. \end{cases}$$

In general it is known that $Z(V, p, T)$ is a rational function.

Theorem 2.3 (Dwork [5]). *$Z(V, p, T)$ is a rational function.*

Remark 2.4. This is true for schemes of finite type over $\mathbb{Z}$. In particular, $Z(V, p, T)$ is a rational function for any open set V (e.g. irreducible part of a character variety).

Hence we can write down $Z(V, p, T)$ as a product of monomials as

$$Z(V, p, T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#V(\mathbb{F}_{p^n})}{n} T^n\right) = \frac{\prod_i (1 - \alpha_i T)}{\prod_j (1 - \beta_j T)}.$$

where $\alpha_i, \beta_j \in \mathbb{C}$ are algebraic numbers. Then we have

$$\#V(\mathbb{F}_{p^n}) = \sum_j \beta_j^n - \sum_i \alpha_i^n \quad (\alpha_i, \beta_j \in \mathbb{C}).$$

Therefore if it is possible to compute $Z(V, p, T)$ then we can obtain $(\#V(\mathbb{F}_{p^n}))_n$. Moreover we can also expect to study the qualitative properties of $(\#V(\mathbb{F}_{p^n}))_n$ like the Weil conjecture for smooth projective varieties over finite fields.

Typical examples of algebraic sets in Algebraic topology are the following two cases:

- The set of $SL_2(\mathbb{C})$-representations of a finitely presented group
- $SL_2(\mathbb{C})$-character variety of a finitely presented group

It is not known whether the set of conjugacy classes of $SL_2(\mathbb{F}_{p^n})$-representations of a finitely presented group is an algebraic set or not. (It is true that the set of conjugacy classes of absolutely irreducible $SL_2(\mathbb{F}_{p^n})$-representations of a finitely presented group is an algebraic set.) I will list up some work in this direction. For detail, see the corresponding references.

- Work of Sink [24].
- Computation of the number of conjugacy classes of $SL_2(\mathbb{F}_q)$-representations of torus knot groups (Li, Xu [16, 17]).
- Computation of the number of conjugacy classes of $SL_2(\mathbb{F}_p)$ (surjective) representations of knots in the Rolfsen's table (Kitano,Suzuki [13]).

2.2 Hasse-Weil zeta function

For given polynomials $f_1, \cdots, f_r \in \mathbb{Z}[X_1, \cdots, X_m]$ its Hasse-Weil zeta function is defined by the product of the local zeta functions as follows:

$$\zeta(V, s) := \zeta(V(f_1, \cdots, f_r), s) := \prod_{p\,:\,\text{prime}} Z(V, p, p^{-s}).$$

$\zeta(V, s)$ converges absolutely in $\mathrm{Re}(s) > \dim V(f_1, \cdots, f_r)$ ([22]). It is conjectured to have meromorphic continuation but it is not known in general ([21]).

Example 2.5 (Riemann zeta function)**.** If $f = X \in \mathbb{Z}[X]$ then as we see in the previous subsection we have $Z(V, p, T) = 1/(1 - T)$. Therefore we have

$$\zeta(V, s) := \prod_{p:\,\text{prime}} (1 - p^{-s})^{-1} = \zeta(s),$$

which is the Riemann zeta function.

In general, if K is an algebraic number field (finite extension field of $\mathbb{Q}$), its ring of integers $\mathcal{O}_K$ is a finitely generated algebra. Hence it is written as $\mathcal{O}_K \xrightarrow{\sim} \mathbb{Z}[X_1, \cdots, X_m]/(f_1, \cdots, f_r)$. Therefore

$$\zeta(V, s) = \zeta(K, s) := \prod_{\mathfrak{p}:\text{non-zero prime}} (1 - N(\mathfrak{p})^{-s})^{-1} : \text{Dedekind zeta function}$$

where $N(\mathfrak{p}) = \#(\mathcal{O}_K/\mathfrak{p})$ is the norm of the prime ideal $\mathfrak{p}$. However the polynomials which generate the kernel is difficult to compute in general. Here we present one well-known case.

Example 2.6 (zeta of cyclotomic polynomial). Let $\Phi_d(X)$ be the d-th cyclotomic polynomial, namely the minimal polynomial of a primitive d-th root ζ_d of unity. For small d, we have

$$\Phi_1(X) = X - 1, \quad \Phi_2(X) = X + 1, \quad \Phi_3(X) = X^2 + X + 1, \quad \Phi_4(X) = X^2 + 1.$$

Let $\mathbb{Q}_d := \mathbb{Q}(\zeta_d)$ be the d-th cyclotomic field. It is well-known that the ring of integers $\mathcal{O}_d$ of $\mathbb{Q}_d$ is equal to $\mathbb{Z}[\zeta_d]$ (cf. [29]). Hence we have $\mathcal{O}_d \xrightarrow{\sim} \mathbb{Z}[X]/(\Phi_d(X))$. Therefore we see that $\zeta(\Phi_d(X), s) = \zeta(\mathbb{Q}_d, s)$. For instance $\zeta(X^2 + 1, s) = \zeta(\mathbb{Q}(\sqrt{-1}), s)$.

3 Hasse-Weil zeta functions of hyperbolic 3-manifolds

3.1 SL_2-character variety

Here we briefly review the definition of the $\mathrm{SL}_2(\mathbb{C})$-character variety. For details, see the original paper [4] or [23].

Let M be an orientable complete hyperbolic 3 manifold with finite volume. Let $X(M)$ be the set of the characters of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(M)$, namely

$$X(M) := \{\text{characters of } \rho : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})\}.$$

The set $X(M)$ is known to be an (affine) algebraic set over $\mathbb{Q}$, that is, it is expressed as the set of zeros of some finite number of polynomials with coefficients in $\mathbb{Q}$. Let $X_0(M)$ be an irreducible component of $X(M)$ which contains the character corresponding to the holonomy representation of M, which is called a canonical component of M. In general it is not known about the dimension of $X(M)$. However the following result is known for the dimension of $X_0(M)$.

Theorem 3.1 (Thurston [25]). *if M is a complete hyperbolic 3 manifold with cusp n then* $\dim X_0(M) = n$.

There is a way to compute the defining polynomials of $X(M)$ from a given group presentation of the fundamental group $\pi_1(M)$ of M. For an concrete example, see for instance [18] for two-bridge knots by Riley's method, and see [7] in the general case.

3.2 Hasse-Weil zeta function of M

Now we define the Hasse-Weil zeta function of a hyperbolic 3 manifold M. Since $X(M)$ is an affine algebraic set over $\mathbb{Q}$ there are finitely many polynomials in $\mathbb{Q}[X_1, \cdots, X_m]$. Thus we can obtain polynomials in $\mathbb{Z}[X_1, \cdots, X_m]$ by multiplying the above polynomials by some positive integer. Let $Z(X(M), p, T)$ be the local zeta function for those polynomials. Then define $\zeta(M, s)$ by the following:

$$\zeta(M, s) := \prod_{p:\ \text{prime}} Z(X(M), p, p^{-s}).$$

Remark 3.2. There is an ambiguity to take the defining polynomials of $X(M)$ with integer coefficients. However we can see that it only affects difference of rational functions in p^{-s} for finitely many primes p. Thus $\zeta(M, s)$ is defined up to rational functions in $\mathbb{Q}(p^{-s})$ for finitely many prime numbers p. In the following examples we will abbreviate these rational functions even for the zeta functions of polynomials with integer coefficients.

3.3 Hasse-Weil zeta functions of the figure 8 knot, two-bridge knots

The description of the $\mathrm{SL}_2(\mathbb{C})$-character variety of the figure 8 knot complement M_8 in S^3 is well-known. It is defined by the polynomial

$$(x^2 - y - 2)(y^2 - (1 + x^2)y + 2x^2 - 1) = 0.$$

The canonical component of M_8 is the elliptic curve defined by $(y^2 - (1 + x^2)y + 2x^2 - 1) = 0$. Its Weierstrass model is $E : Y^2 = X^3 - 2X + 1$. This is an elliptic curve over $\mathbb{Q}$ whose conductor is 40. The curve $x^2 - y - 2 = 0$ corresponds to the points of the reducible characters. It would depend on the situation whether we should consider the zeta function of the whole character variety or just the canonical component. Here we present the zeta function of the irreducible part of $X(M_8)$.

3.3.1 Hasse-Weil zeta function of figure 8 knot

Theorem 3.3 (H. [9]).

$$\zeta(M_8, s) = \frac{\zeta(E, s)}{\zeta(s)\zeta(\mathbb{Q}(\sqrt{5}), s)},$$

At present it is difficult to study the Hasse-Weil zeta functions of algebraic varieties in general. Except the dimension 0 case (Dedekind zeta case), elliptic curves over $\mathbb{Q}$ are the almost only case we can study some properties of the zeta functions in the general setting. Since the above elliptic curve has analytic rank 0, we can also study its special values.

- If we consider the following completed function

$$\xi(M_8, s) := \frac{4\pi^{(3s/2)+1}}{(10\sqrt{2})^s\Gamma(s/2)^3} \times \zeta(M_8, s)$$

  it has a functional equation $\xi(M_8, 2-s)\xi(2-s)\xi(\mathbb{Q}(\sqrt{5}), 2-s) = -\xi(M_8, s)\xi(s)\xi(\mathbb{Q}(\sqrt{5}), s)$.

- The special value of $\xi(M_8, s)$ at $s = 1$ is computed as

$$\lim_{s\to 1} \frac{\xi(M_8, s)}{s-1} = -\frac{\mathrm{AGM}(\varphi, \varphi - 1)}{\sqrt{10}\log(\varphi)}.$$

  Here $\varphi = (\sqrt{5} + 1)/2$ and $\mathrm{AGM}(\varphi, \varphi - 1)$ is the arithmetic geometric mean.

We remark that here the Hasse-Weil zeta function $\zeta(E, s)$ means the Hasse-Weil zeta function of the projective model of the affine curve $E : Y^2 = X^3 - 2X + 1$.

For any Laurent polynomial $P = P(t_1, \cdots, t_n) \in \mathbb{C}[t_1^{\pm 1}, \cdots, t_n^{\pm 1}]$ the Mahler measure of P is defined by

$$m(P) := \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi t_1 \sqrt{-1}}, \cdots, e^{2\pi t_n \sqrt{-1}})| dt_1 \cdots dt_n.$$

There are some research trying to find geometric interpretation of the Mahler measures of polynomial invariants in Algebraic Topology. For instance see [1]. Since $\zeta(M_8, s)$ has a functional equation between s and $2 - s$, special values at $s = 1, 2$ should be the most interesting ones. We have some interpretation of the special values of $\zeta(M_8, s)$ at $s = 1, 2$ in terms of the Mahler measures of certain polynomials.

- $\log(\varphi^2) = m(\Delta_{\mathcal{K}}(T))$, where $\Delta_{\mathcal{K}}(T)$ is the Alexander polynomial of $\mathcal{K}$.
- $\mathrm{AGM}(\varphi, \varphi - 1) = \frac{1}{2}\left(\frac{d}{dk} m(P_k)(\sqrt{5})\right)^{-1}$. Here $P_k := x + \frac{1}{x} + y + \frac{1}{y} - 4k$ is a family of elliptic curves for $4k \neq 0, 1$.

Remark 3.4. The special Value at $s = 2$ is as follows.

$$\lim_{s\to 2}(s-2)\xi(M_8, s) = \frac{75}{2\sqrt{5}\pi^2 \mathcal{L}(E, 2)}.$$

Here $\mathcal{L}(E, s)$ is the completed L-series of $E/\mathbb{Q}$. It is numerically observed by Rodriguez-Villegas ([27], TABLE 4) that $\mathcal{L}_{E/\mathbb{Q}}(2)$ would be equal to $m(P_{\sqrt{-4}/4})$. In fact this has been confirmed by Mellit ([20]).

3.3.2 Two-bridge knots case

We would like to obtain similar results to the figure 8 knot case for more general families of knots. Hence it is natural to consider the two-bridge knots case for this purpose.

It is known that the $\mathrm{SL}_2(\mathbb{C})$-character varieties of the two-bridge knots have dimension 1. Still it is not so easy to study the algebraic geometric properties of their character varieties.

Macasieb, Petersen, Van Luijk [18] have studied the genus of $\mathrm{SL}_2(\mathbb{C})$-character curves of certain family of hyperbolic two-bridge knots which contains all the twist knots. In that family of two-bridge knots, only two canonical curves have genus 1, elliptic curves.

Example 3.5. Up to (a, b) $(a, b < 50)$, In the 5 cases $(5, 2) = 4_1, (15, 4) = 7_4, (21, 8) = 7_7, (27, 8) = 8_{11}, (45, 14) = 10_{21}$ canonical curves are elliptic curves.

Question 3.6. *Are there finitely many elliptic curves appeared as canonical components of* $\mathrm{SL}_2(\mathbb{C})$*-character varieties of hyperbolic two-bridge knots?*

3.4 Hasse-Weil zeta function of arithmetic two-bridge link

An arithmetic hyperbolic 3-manifold is a special class of hyperbolic 3-manifolds. They are relatively simple among hyperbolic 3-manifolds, since their commensurable classes are determined by the pair of the invariant trace fields and the invariant quaternion algebras associated with the arithmetic 3-manifolds. It is also known that the smallest volume hyperbolic 3-manifolds with cusp 0, 1, 2 are arithmetic. For details, see [19].

In what follows, we use the notation 'arithmetic' in somewhat strict sense. Namely, a hyperbolic 3 manifold M is arithmetic when the image of the holonomy representation $\rho_M : \pi_1(M) \to \mathrm{SL}_2(\mathbb{C})$ is in $\mathrm{SL}_2(\mathcal{O})$ up to conjugacy, where $\mathcal{O}$ is the ring of integers of some number field.

Theorem 3.7 (Gehring, Maclachlan, Martin [6]). *Arithmetic two-bridge links in the* 3*-sphere are the figure* 8 *knot, the Whitehead link* $5_1^2 = (8, 3)$, $6_2^2 = (10, 3)$, $6_3^2 = (12, 5)$.

Here we consider the canonical components of $\mathrm{SL}_2(\mathbb{C})$-character varieties of the Whitehead link, 6_2^2, 6_3^2.

Defining polynomials of the canonical components of the Whitehead link, 6_2^2, 6_3^2 in $\mathbb{C}^3$ are

$$\begin{aligned} f_0 &:= z^3 - xyz^2 + (x^2 + y^2 - 2)z - xy, \\ f_1 &:= z^4 - xyz^3 + (x^2 + y^2 - 3)z^2 - xyz + 1, \\ f_2 &:= z^3 - xyz^2 + (x^2 + y^2 - 1)z - xy. \end{aligned}$$

These surfaces $V(f_i) \subset \mathbb{C}^3$ are smooth affine surfaces.

Put

$$\mathbb{P}^2 \times \mathbb{P}^1 := \left\{(x : y : u,\ z : w) \,\middle|\, (x : y : u) \in \mathbb{P}^2,\ (z : w) \in \mathbb{P}^1\right\}.$$

Consider the compactification $X(F_i) \subset \mathbb{P}^2 \times \mathbb{P}^1$ of these surfaces in $\mathbb{P}^2 \times \mathbb{P}^1$. Here

$$\begin{aligned} F_0 &:= u^2z^3 - xyz^2w + (x^2 + y^2 - 2u^2)zw^2 - xyw^3, \\ F_1 &:= u^2z^4 - xyz^3w + (x^2 + y^2 - 3u^2)z^2w^2 - xyzw^3 + u^2w^4, \\ F_2 &:= u^2z^3 - xyz^2w + (x^2 + y^2 - u^2)zw^2 - xyw^3. \end{aligned}$$

The surfaces $X(F_i) \subset \mathbb{P}^2 \times \mathbb{P}^1$ are considered as (singular) conic bundles over $\mathbb{P}^1$ by $(x : y : u,\ z : w) \mapsto (z : w)$ (Landes [14, 15], H. [8]), which enables us to compute the number of $\mathbb{F}_q$-rational points of F_i.

Theorem 3.8 (H. [10]). *Let X_0, X_1, X_2 be the canonical components of the Whitehead link $5_1^2 = (8,3)$, $6_2^2 = (10,3)$, $6_3^2 = (12,5)$. Then we have*

$$\zeta(X_0, s) = \zeta(\mathbb{Q}(\sqrt{2}), s-1)\zeta(s)^2\zeta(s-1)^2\zeta(s-2).$$
$$\zeta(X_1, s) = \zeta(\mathbb{Q}(\sqrt{5}), s-1)^2\zeta(s)^3\zeta(s-2).$$
$$\zeta(X_2, s) = \zeta(s)^2\zeta(s-2).$$

Remark 3.9. Note that
$\mathbb{Q}(\sqrt{2}), \mathbb{Q}(\sqrt{5}), \mathbb{Q}$ for $\zeta(X_i, s) = \mathbb{Q}\begin{pmatrix} \text{all the } \mathbb{P}^1\text{-coordinates} \\ \text{of the degenerate fibers of } X_i \end{pmatrix}$. They are different from the trace fields (invariant trace fields) $\mathbb{Q}(\sqrt{-1})$, $\mathbb{Q}(\sqrt{-3})$ and $\mathbb{Q}(\sqrt{-7})$ of 5_1^2, 6_2^2 and 6_3^2.

Remark 3.10. X_i is birationally equivalent to $\mathbb{P}^2$. However the zeta functions of $\mathbb{A}^2_{\mathbb{Q}}$, $\mathbb{P}^2_{\mathbb{Q}}$ are $\zeta(\mathbb{A}^2_{\mathbb{Q}}, s) = \zeta(s-2)$, $\zeta(\mathbb{P}^2_{\mathbb{Q}}, s) = \zeta(s)\zeta(s-1)\zeta(s-2)$.

Recently the following result was obtained, which contains The Whitehead link, 6_2^2 cases.

Theorem 3.11 (Tran [26]). *The $\mathrm{SL}_2(\mathbb{C})$-character varieties of the two-bridge link $(2m, 3)$ $(m \neq 3)$ have 2 irreducible components. Canonical components are defined in terms of Chebyshev polynomials. Their compactification have conic bundle structure.*

In general, the canonical components of the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic two-bridge link do not have a conic bundle structure over $\mathbb{P}^1$ (For examples, see [14, 15]). However we might expect the following.

Question 3.12. *Does the canonical component of the $\mathrm{SL}_2(\mathbb{C})$-character variety of a hyperbolic two-bridge link have a fibered structure over $\mathbb{P}^1$?*

3.5 Hasse-Weil zeta function of closed arithmetic 3 manifold

Here we present some examples of the Hasse-Weil zeta functions of the irreducible part of the $\mathrm{SL}_2(\mathbb{C})$-character varieties of closed arithmetic 3-manifolds with small volumes.

In the table below $X(M)_{\mathrm{Irr}}(\mathbb{C})$ means the open subset of the $\mathrm{SL}_2(\mathbb{C})$-character variety of M consisting of irreducible characters, K_M is the trace field $\mathbb{Q}(\mathrm{Tr}\,\rho_M(\pi_1(M))$ of M, where ρ_M denotes the holonomy representation of M. Note that in the examples below the character varieties $X(M)$ have dimension 0.

M	def.poly. of $X(M)_{\mathrm{Irr}}(\mathbb{C})$	$\zeta(M,s)$
Weeks	$T^3 - T - 1$	$\zeta(K_M, s)$
Meyerhoff	$T^4 - 3T^3 + T^2 + 3T - 1$	$\zeta(K_M, s)$
m010 (-1,2)	$T^4 - 2T^2 + 4$	$\zeta(K_M, s)$
m003 (-4,3)	$T^4 - T^3 - 2T^2 + 2T + 1$	$\zeta(K_M, s)$
m004 (6,1)	$T^6 - 7T^4 + 14T^2 - 4$	$\zeta(K_M, s)$
m003 (-3,4)	$T^6 + T^4 - 1$	$\zeta(K_M, s)$

The Hasse-Weil zeta functions $\zeta(X(M)_{\mathrm{Irr}}, s)$ are equal to the Dedekind zeta functions of the trace fields of M (up to $\mathbb{Q}(p^{-s})$ for finitely many primes p). Thus we may expect the following.

Question 3.13. *Let M be an (arithmetic) closed 3 manifold and K_M the trace field of M. Then*

$$\zeta(M,s) = \zeta(K_M, s)?$$

Moreover,

$$\zeta(M,s) = \zeta(M', s) \Longleftrightarrow K_M \xrightarrow{\sim} K_{M'}?$$

The second statement might be true since this is true for invariant trace fields as follows.

Theorem 3.14 (Chinburg, Hamilton, Long, Reid [2]). *Let K, K' be number fields having only one complex place. Then K and K' are isomorphic if and only if $\zeta(K,s) = \zeta(K', s)$.*

4 Hasse-Weil zeta function of A-polynomials of torus knots

4.1 A-polynomial of knots in S^3

Here we review some properties of the A-polynomials of knots which are needed in this note. For details, see the original paper [3].

Let $\mathcal{K}$ be a knot in S^3 and let $\lambda, \mu \in \pi_1(S^3 \setminus \mathcal{K})$ be the canonical longitude and meridian of $\mathcal{K}$. Let

$$R(\mathcal{K}) = \left\{\rho : \pi_1(S^3 \setminus \mathcal{K}) \to \mathrm{SL}_2(\mathbb{C})\right\}$$

be the set of $\mathrm{SL}_2(\mathbb{C})$-representations of $\pi_1(S^3 \setminus \mathcal{K})$ and

$$R_U = \{\rho \in R(\mathcal{K}) \mid \rho(\lambda), \rho(\mu) : \text{ upper triangular } \}$$

the subset of $R(\mathcal{K})$ consisting of upper triangular matrices in $R(\mathcal{K})$. Then define the eigenvalue map ξ by

$$\xi : R_U \to \mathbb{C}^\times \times \mathbb{C}^\times, \quad \rho \mapsto (a_\rho(\lambda), a_\rho(\mu))$$

if
$$\rho(\lambda)=\begin{pmatrix} a_\rho(\lambda) & b_\rho(\lambda) \\ 0 & a_\rho(\lambda)^{-1}\end{pmatrix},\quad \rho(\mu)=\begin{pmatrix} a_\rho(\mu) & b_\rho(\mu) \\ 0 & a_\rho(\mu)^{-1}\end{pmatrix}.$$
Then we can obtain an algebraic curve in $\mathbb{C}^2$ by considering
$$\mathbb{C}^2 \supset \bigcup_C \overline{\xi(C)}.$$
Here C runs through all the irreducible components of R_U whose closures $\overline{\xi(C)}$ in $\mathbb{C}^2$ are curves. Since this is a plane curve, it is defined by a (reduced) polynomial $A_{\mathcal{K}}(L,M) \in \mathbb{C}[L,M]$, namely
$$\mathbb{C}^2 \supset \bigcup_C \overline{\xi(C)} = V(A_{\mathcal{K}}(L,M)).$$
This polynomial is called the A-polynomial of $\mathcal{K}$. By definition it is defined up to constant, multiplicity. Here are some properties of the A-polynomials of knots in S^3.

- $A_{\mathcal{K}}(L,M) \in \mathbb{Z}[L,M]$,
- $A_{\bigcirc}(L,M) = L-1$,
- $L-1 \mid A_{\mathcal{K}}(L,M)$,
- $\mathcal{K}$: non-trivial $\Longrightarrow A_{\mathcal{K}}(L,M) \neq L-1$.

4.2 Hasse-Weil zeta functions of A-polynomials of torus knots

Let a,b be positive integers such that $(a,b)=1$. The general form of the A-polynomials of (a,b)-torus knots $T(a,b)$ are well-known as follows.
$$A_{a,b}(L,M) := A_{T(a,b)}(L,M) = \begin{cases} (L-1)\left(-1+(LM^{ab})^2\right), & \text{if } a,b>2, \\ (L-1)(1+LM^{2(2m+1)}), & \text{if } (a,b)=(2,2m+1). \end{cases}$$
Then the Hasse-Weil zeta functions of $A_{a,b}(L,M)$ are expressed as follows.

Theorem 4.1 (H.-Terashima [11]). *Up to rational functions in $\mathbb{Q}(\{p^{-s}\}_{p|2ab})$ the zeta function $\zeta(A_{a,b}(L,M),s)$ is equal to*
$$\begin{cases} \left(\displaystyle\prod_{d|2ab}\frac{1}{\zeta(\mathbb{Q}_d,s)}\right)\dfrac{\zeta(s-1)^3}{\zeta(s)^2}, & \text{if } a,\ b>2, \\ \left(\displaystyle\prod_{d|2ab,\ d\nmid ab}\frac{1}{\zeta(\mathbb{Q}_d,s)}\right)\dfrac{\zeta(s-1)^2}{\zeta(s)}, & \text{if } (a,b)=(2,2m+1). \end{cases}$$

Example 4.2 (trivial knot). $\zeta(A_{\bigcirc}(L,M)) = \zeta(s-1)$.

Example 4.3 (trefoil knot). $\zeta(A_{2,3}(L,M),s) = \dfrac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4,s)^{-1}\zeta(\mathbb{Q}_{12},s)^{-1}$.

Example 4.4. $\zeta(A_{2,5}(L,M),s) = \dfrac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4,s)^{-1}\zeta(\mathbb{Q}_{20},s)^{-1}$.

In general the A-polynomial of a knot can be divided by $L-1$. In fact it is clear from the definition of the zeta function that $\zeta(A_{a,b}(L,M),s)$ is divided by $\zeta(L-1,s) = \zeta(s-1)$. Therefore considering reduced A-polynomials divided by $L-1$ might be better in certain cases. However when we consider the zeta function this is not the case. For each component of the A-polynomial of a torus knot its zeta function has the following form:

$$\zeta(L-1,s) = \zeta(s-1), \quad \zeta(-1+(LM^{ab})^2,s) = \frac{\zeta(s-1)^2}{\zeta(s)^2}, \quad \zeta(1+LM^{2(2m+1)},s) = \zeta(s-1)/\zeta(s).$$

Therefore we have to consider the intersection of two components of the A-polynomial to retrieve essential information (in the torus knot case, invariants with respect to a,b).

4.3 Characters of the minimal model

Here we explain a relation between the description of the Hasse-Weil zeta function of the A-polynomial of a torus knot and the Kashaev invariant of the torus knot.

Let a,b be coprime positive integers and put

$$c(a,b) = 1 - \frac{6(a-b)^2}{ab}, \quad h^{a,b}_{n,m} = \frac{(nb-ma)^2-(a-b)^2}{4ab}.$$

Minimal models are a series of infinite dimensional representations of the Virasoro algebra, namely those are irreducible highest weight representations of the Virasoro algebra with conformal weight $h^{a,b}_{n,m}$ and central charge $c(a,b)$ (where $1 \le n \le a-1, 1 \le m \le b-1$). For details we refer to [28], Chapter 7.

Especially the normalized character $\mathrm{ch}^{a,b}_{n,m}(\tau)$ of a minimal model $\mathcal{M}(a,b)$ has a presentation in terms of the Dedekind η-function as follows (cf. [12], [28]):

$$\mathrm{ch}^{a,b}_{n,m}(\tau) = \frac{\Phi^{a,b}_{n,m}(\tau)}{\eta(\tau)}, \quad \eta(\tau) = q^{\frac{1}{24}}\prod_{k\ge1}(1-q^k).$$

Here $q = e^{2\pi i\tau}$. Note that

$$\mathrm{ch}^{a,b}_{n,m}(\tau) = \mathrm{ch}^{a,b}_{a-n,b-m}(\tau) = \mathrm{ch}^{b,a}_{m,n}(\tau) = \mathrm{ch}^{b,a}_{b-m,a-n}(\tau).$$

Therefore there are $(a-1)(b-1)/2$ characters for the minimal model $\mathcal{M}(a,b)$. Here

$$\Phi^{a,b}_{n,m}(\tau) = \sum_{k=0}^{\infty}\chi^{(n,m)}_{2ab}(k)q^{\frac{k^2}{4ab}}.$$

The 'character' $\chi_{2ab}^{(n,m)} : \mathbb{Z} \to \mathbb{C}$ is defined by

$$\chi_{2ab}^{(n,m)}(k) = \begin{cases} 1, & k \equiv \pm(nb - ma) \pmod{2ab}, \\ -1, & k \equiv \pm(nb + ma) \pmod{2ab}, \\ 0, & \text{otherwise.} \end{cases}$$

Especially we note that $\chi_{2ab}^{(n,m)}(0) = 0$.

Remark 4.5. Consider the normalized Alexander polynomial of the torus knot $T(a, b)$

$$\Delta_{a,b}(T^2) = \frac{(T^{ab} - T^{-ab})(T - T^{-1})}{(T^a - T^{-a})(T^b - T^{-b})}.$$

(Here we consider $\Delta_{a,b}(T^2)$ just for convenience.) Then its 'inverse' has the following power series expansion:

$$\frac{T - T^{-1}}{\Delta_{a,b}(T^2)} = \frac{(T^a - T^{-a})(T^b - T^{-b})}{(T^{ab} - T^{-ab})} = \sum_{k \geq 0} \chi_{2ab}^{(a-1,1)}(k) T^{-k}.$$

4.4 Relation with Quantum invariant

Consider the 'Eichler integral' of $\Phi_{n,m}^{a,b}(\tau)$

$$\tilde{\Phi}_{n,m}^{a,b}(\tau) = -\frac{1}{2} \sum_{k=0}^{\infty} k \chi_{2ab}^{(n,m)}(k) q^{\frac{k^2}{4ab}}.$$

When $(n, m) = (a - 1, 1)$, the following relation with the Kashaev invariant $\langle T(a, b) \rangle_N$ of $T(a, b)$ is known.

Theorem 4.6 (Hikami-Kirillov [12]). $\langle T(a, b) \rangle_N = \tilde{\Phi}_{a-1,1}^{a,b}(1/N) \times \exp\left(\frac{(ab - a - b)^2}{2abN} \pi i\right)$.

We would like to see a relation between the function $\tilde{\Phi}_{a-1,1}^{a,b}(1/N)$ and the description of $\zeta(A_{a,b}(L, M), s)$.

The asymptotic expansion of $\tilde{\Phi}_{a-1,1}^{a,b}(1/N)$ for $N \to \infty$ is as follows ([12]):

$$\tilde{\Phi}_{n,m}^{a,b}(1/N) + (-iN)^{\frac{3}{2}}\text{-term} \sim \sum_{k=0}^{\infty} \frac{T^{n,m}(k)}{k!} \left(\frac{\pi}{2abiN}\right)^k,$$

where

$$T^{n,m}(k) = \frac{1}{2}(-1)^{k+1} L(-2k - 1, \chi_{2ab}^{(n,m)})$$

and $L(s, \chi_{2ab}^{(n,m)})$ is the L-function of $\chi_{2ab}^{(n,m)}$ defined by the power series

$$L(s, \chi_{2ab}^{(n,m)}) = \sum_{k \geq 1} \frac{\chi_{2ab}^{(n,m)}(k)}{k^s}$$

for $\mathrm{Re}(s) > 1$, which has a meromorphic continuation to $\mathbb{C}$.

Remark 4.7. Consider $s = \log T$ for $L\left(s, \chi_{2ab}^{(n,m)}\right)$. Then

$$\sum_{k\geq 1} \chi_{2ab}^{(n,m)}(k) T^{-\log k} = L\left(\log T, \chi_{2ab}^{(n,m)}\right)$$

When we consider $(n, m) = (a-1, 1)$, essentially $L\left(s, \chi_{2ab}^{(n,m)}\right)$ corresponds to the the previous power series expansion of the Alexander polynomial $\Delta_{a,b}(T^2)$.

From the Fourier transform on finite abelian groups $\chi_{2ab}^{(n,m)}$ is written in terms of the Dirichlet characters of modulo $2ab$ as follows:

$$\chi_{2ab}^{(n,m)} = \frac{1}{\phi(2ab)} \sum_{\chi:\text{ even}} c_\chi(a,b,n,m)\chi,$$

where $\phi(2ab) = \#(\mathbb{Z}/2ab\mathbb{Z})^\times$ is the Euler function and χ runs through all the even Dirichlet characters modulo $2ab$ (that is, $\chi(-1) = 1$), and

$$c_\chi(a,b,n,m) = \overline{2(\chi(nb-ma) - \chi(nb+ma))}.$$

Therefore its L-function is written in terms of the L-functions of the even Dirichlet characters modulo $2ab$

$$L\left(s, \chi_{2ab}^{(n,m)}\right) = \frac{1}{\phi(2ab)} \sum_{\chi:\text{ even}} c_\chi(a,b,n,m) L(s,\chi).$$

In general the Dedekind zeta functions of abelian number fields are written as the product of Dirichlet L-functions. Especially, in the cyclotomic field case, we have

$$\zeta(\mathbb{Q}_d, s) = \prod_\chi L(s,\chi),$$

where χ runs through all the Dirichlet characters modulo d. Therefore

$$L\left(s, \chi_{2ab}^{(n,m)}\right) = \frac{1}{\phi(2ab)} \sum_{\chi:\text{ even}} c_\chi(a,b,n,m) L(s,\chi).$$

Example 4.8. Consider the $(2,3)$-torus knot case. In this case, there is only one character $\chi_{12}^{(1,1)} = \chi_{12}^{(1,2)}$, which is the unique even Dirichlet character modulo 12. The zeta function $\zeta(A_{2,3}(L,M), s)$ is expressed as

$$\zeta(A_{2,3}(L,M), s) = \frac{\zeta(s-1)^2}{\zeta(s)} \times \zeta(\mathbb{Q}_4, s)^{-1} \zeta(\mathbb{Q}_{12}, s)^{-1}.$$

Since $(\mathbb{Z}/12\mathbb{Z})^\times$ has 4 characters, the Dedekind zeta function $\zeta(\mathbb{Q}_{12}, s)$ is decomposed into the product of the Dirichlet L-functions

$$\zeta(\mathbb{Q}_{12}, s) = \zeta(s) L(s, \chi_{12}^{(1,1)}) L(s, \chi_2) L(s, \chi_3),$$

where χ_2, χ_3 are the other two odd Dirichlet characters of $(\mathbb{Z}/12\mathbb{Z})^\times$. The L-function $L(s, \chi_{12}^{(1,1)})$ essentially corresponds to the Alexander polynomial $\Delta_{2,3}(T^2)$.

Question 4.9. *Is there a topological interpretation for* $\tilde{\Phi}^{a,b}_{n,m}(\tau)$ *and* $L\left(s, \chi^{(n,m)}_{2ab}\right)$ *for* $(n, m) \neq (a-1, 1)$*?*

Question 4.10. *Is there a direct relation between* $\langle T(a, b)\rangle_N$ *(or colored Jones polynomial of* $T(a, b)$*) and* $L\left(s, \chi^{(n,m)}_{2ab}\right)$ *(or other components of* $\zeta(A_{a,b}(L, M), s)$*)?*

Acknowledgments

I would like to thank Professor Tomotada Ohtsuki for inviting me the workshop "Intelligence of Low-dimensional Topology 2014" and giving the opportunity to write this article. The author is supported by the JSPS Fellowships for Young Scientists.

References

[1] D. W. Boyd, *Mahler's measure and invariants of hyperbolic manifolds*, Number theory for the millennium, I (Urbana, IL, 2000), 2002, 127–143.

[2] T. Chinburg, E. Hamilton, D. D. Long, and A. W. Reid, *Geodesics and commensurability classes of arithmetic hyperbolic 3-manifolds*, Duke Math. J. **145** (2008), 25–44.

[3] D. Cooper, M. Culler, H. Gillet, D. D. Long, and P. B. Shalen, *Plane curves associated to character varieties of* 3*-manifolds*, Invent. Math. **118** (1994), no. 1, 47–84. MR MR1288467 (95g:57029)

[4] M. Culler and P. B. Shalen, *Varieties of group representations and splittings of* 3*-manifolds*, Ann. of Math. (2) **117** (1983), 109–146.

[5] B. Dwork, *On the rationality of the zeta function of an algebraic variety*, Amer. J. Math. **82** (1960), 631–648.

[6] F. W. Gehring, C. Maclachlan, and G. J. Martin, *Two-generator arithmetic Kleinian groups. II*, Bull. London Math. Soc. **30** (1998), 258–266.

[7] F. González-Acuña and J. M. Montesinos-Amilibia, *On the character variety of group representations in* SL(2, **C**) *and* PSL(2, **C**), Math. Z. **214** (1993), 627–652.

[8] S. Harada, *Canonical components of character varieties of arithmetic two bridge link complements*, preprint, http://arxiv.org/abs/1112.3441.

[9] ———, *Hasse-Weil zeta function of absolutely irreducible* SL_2*-representations of the figure 8 knot group*, Proc. Amer. Math. Soc. **139** (2011), 3115–3125.

[10] ———, *Hasse-Weil zeta functions of* SL_2*-character varieties of arithmetic two-bridge link complements*, Finite Fields Appl. **27** (2014), 115–129.

[11] S. Harada and Y. Terashima, *Hasse-Weil zeta functions of a-polynomials of torus knots*, in preparation.

[12] K. Hikami and A. N. Kirillov, *Torus knot and minimal model*, Phys. Lett. B **575** (2003), 343–348.

[13] T. Kitano and M. Suzuki, *On the number of* $SL(2;\mathbb{Z}/p\mathbb{Z})$*-representations of knot groups*, J. Knot Theory Ramifications **21** (2012), 1250003, 18.

[14] E. Landes, *On the canonical components of character varieties of hyperbolic 2-bridge link complements*, Available at http://repositories.lib.utexas.edu/bitstream/handle/2152/ETD-UT-2011-08-2877/LANDES-DISSERTATION.pdf?sequence=1.

[15] ———, *Identifying the canonical component for the Whitehead link*, Math. Res. Lett. **18** (2011), 715–731.

[16] W. Li and L. Xu, *Counting* $\mathrm{SL}_2(\mathbf{F}_{2^s})$ *representations of torus knot groups*, Acta Math. Sin. (Engl. Ser.) **19** (2003), 233–244.

[17] ———, *Counting* $\mathrm{SL}_2(\mathbf{F}_q)$*-representations of torus knot groups*, J. Knot Theory Ramifications **13** (2004), 401–426.

[18] M. L. Macasieb, K. L. Petersen, and R. M. van Luijk, *On character varieties of two-bridge knot groups*, Proc. Lond. Math. Soc. (3) **103** (2011), 473–507.

[19] C. Maclachlan and A. W. Reid, *The arithmetic of hyperbolic 3-manifolds*, Grad. Texts in Math., **219**, Springer-Verlag, 2003.

[20] A. Mellit, *Elliptic dilogarithms and parallel lines*, preprint.

[21] J.-P. Serre, *Facteurs locaux des fonctions ζeta des variétés algébriques (définitions et conjectures)*, Séminaire Delange-Pisot-Poitou: 1969/70.

[22] ———, *Zeta and L functions*, Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963), 1965, 82–92.

[23] P. B. Shalen, *Representations of 3-manifold groups*, Handbook of geometric topology, North-Holland, Amsterdam, 2002, 955–1044.

[24] J. M. Sink, *A zeta-function for a knot using* $\mathrm{SL}_2(\mathbb{F}_{p^s})$ *representations*, Knots in Hellas '98 (Delphi), Ser. Knots Everything, **24**, World Sci. Publishing, 2000, 452–470.

[25] W. P. Thurston, *Three-dimensional geometry and topology. Vol. 1*, Princeton Math. Series, **35**, Princeton University Press, 1997.

[26] A. T. Tran, *Character varieties of some families of links*, preprint, http://arxiv.org/abs/1403.6800.

[27] F. R. Villegas, *Modular Mahler measures. I,* Topics in number theory (University Park, PA, 1997), Math. Appl., **467**, Kluwer Acad. Publ., 1999, 17–48.

[28] M. Wakimoto, *Lectures on infinite-dimensional Lie algebra*, World Scientific Publishing, 2001.

[29] L. C. Washington, *Introduction to cyclotomic fields*, second ed., Grad. Texts in Math., **83**, Springer-Verlag, 1997.

Graduate School of Information Science and Engineering
Tokyo Institute of Technology
Tokyo 152-8550
JAPAN
E-mail address: `harada.s.al@m.titech.ac.jp`

東京工業大学・情報理工学研究科　原田　新也